Problem 1.37

Consider the equation $y'' + 2y'/x + y^n = 0$ [$y(\infty) = 0$] for $n \neq 0, 1$. Using the methods of Sec. 1.7 show that the equation is soluble in terms of elementary functions when $n = 5$ and solve it.

Solution

If $n = 5$, then the ODE we have to solve is

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^5 = 0. \tag{1}$$

Make the following scale transformations.

$$\begin{aligned} x &\to ax \\ y &\to a^p y \end{aligned}$$

If there is a value of p that leaves the ODE unchanged, then it is said to be scale invariant, and progress can be made in solving it.

$$\frac{d^2(a^p y)}{d(ax)^2} + \frac{2}{(ax)}\frac{d(a^p y)}{d(ax)} + (a^p y)^5 = 0$$

Pull the constants out of the derivative and separate the a terms from x and y.

$$\frac{a^p}{a^2}\frac{d^2y}{dx^2} + \frac{2}{x}\frac{a^p}{a^2}\frac{dy}{dx} + a^{5p}y^5 = 0$$

Combine the a terms.

$$a^{p-2}\frac{d^2y}{dx^2} + a^{p-2}\frac{2}{x}\frac{dy}{dx} + a^{5p}y^5 = 0$$

Notice that if $p - 2 = 5p$ or $p = -1/2$, then the a terms cancel out and we're back to equation (1). Thus, the ODE is scale invariant under the transformation, $x \to ax$, $y \to a^{-1/2}y$. This means we can make the substitution,

$$y(x) = x^{-1/2}u(x),$$

to make the ODE equidimensional. Take the derivative to find out what y' is in terms of u.

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2}u + x^{-1/2}\frac{du}{dx}$$

Take another derivative to find out what y'' is in terms of u.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{3}{4}x^{-5/2}u - \frac{1}{2}x^{-3/2}\frac{du}{dx} - \frac{1}{2}x^{-3/2}\frac{du}{dx} + x^{-1/2}\frac{d^2u}{dx^2} \\ &= \frac{3}{4}x^{-5/2}u - x^{-3/2}\frac{du}{dx} + x^{-1/2}\frac{d^2u}{dx^2} \end{aligned}$$

Make these substitutions in equation (1).

$$\frac{3}{4}x^{-5/2}u - x^{-3/2}\frac{du}{dx} + x^{-1/2}\frac{d^2u}{dx^2} + \frac{2}{x}\left(-\frac{1}{2}x^{-3/2}u + x^{-1/2}\frac{du}{dx}\right) + (x^{-1/2}u)^5 = 0$$

Distribute $2/x$.

$$\frac{3}{4}x^{-5/2}u - x^{-3/2}\frac{du}{dx} + x^{-1/2}\frac{d^2u}{dx^2} - x^{-5/2}u + 2x^{-3/2}\frac{du}{dx} + x^{-5/2}u^5 = 0$$

Combine like-terms.

$$x^{-1/2}\frac{d^2u}{dx^2} + x^{-3/2}\frac{du}{dx} - \frac{1}{4}x^{-5/2}u + x^{-5/2}u^5 = 0$$

Multiply both sides by $x^{5/2}$.

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - \frac{1}{4}u + u^5 = 0$$

We now have an equidimensional ODE. We can make the exponential substitution, $x = e^t$, to eliminate the independent variable and hence make the ODE autonomous. Doing so means we'll have to write expressions for $u'(x)$ and $u''(x)$ in terms of t using the chain rule.

$$\begin{aligned}
x = e^t \quad &\rightarrow \quad \frac{dx}{dt} = e^t \\
\frac{du}{dt} &= \frac{du}{dx}\frac{dx}{dt} = x\frac{du}{dx} \\
\frac{d^2u}{dt^2} &= \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(x\frac{du}{dx}\right) = \frac{dx}{dt}\frac{d}{dx}\left(x\frac{du}{dx}\right) = x\left(\frac{du}{dx} + x\frac{d^2u}{dx^2}\right) \\
&= x\frac{du}{dx} + x^2\frac{d^2u}{dx^2} = \frac{du}{dt} + x^2\frac{d^2u}{dx^2} \quad \rightarrow \quad \frac{d^2u}{dt^2} - \frac{du}{dt} = x^2\frac{d^2u}{dx^2}
\end{aligned}$$

Plug these expressions into the ODE.

$$\frac{d^2u}{dt^2} - \cancel{\frac{du}{dt}} + \cancel{\frac{du}{dt}} - \frac{1}{4}u + u^5 = 0$$

We now have an autonomous equation.

$$\frac{d^2u}{dt^2} - \frac{1}{4}u + u^5 = 0$$

Reduce the order of the ODE by making the substitution,

$$\begin{aligned}
\frac{du}{dt} &= w(u) \\
\frac{d^2u}{dt^2} &= \frac{dw(u)}{dt} = \frac{dw}{du}\frac{du}{dt} = \frac{dw}{du}w(u).
\end{aligned}$$

Plugging these expressions into the ODE yields the nonlinear first-order equation,

$$\frac{dw}{du}w - \frac{1}{4}u + u^5 = 0,$$

which can be solved by separation of variables. Bring the last two terms to the right side.

$$w\frac{dw}{du} = \frac{1}{4}u - u^5$$

Separate variables.

$$w\,dw = \left(\frac{1}{4}u - u^5\right)du$$

Integrate both sides.

$$\frac{1}{2}w^2 = \frac{1}{8}u^2 - \frac{1}{6}u^6 + A$$

Multiply both sides by 2.

$$w^2 = \frac{1}{4}u^2 - \frac{1}{3}u^6 + 2A$$

Now that the integration is done, change back to du/dt.

$$\left(\frac{du}{dt}\right)^2 = \frac{1}{4}u^2 - \frac{1}{3}u^6 + 2A$$

Unfortunately, I can't see how to solve this equation explicitly for u. An implicit solution can be obtained with separation of variables, though. Take the square root of both sides first.

$$\frac{du}{dt} = \sqrt{\frac{1}{4}u^2 - \frac{1}{3}u^6 + 2A}$$

Then separate variables.

$$\frac{du}{\sqrt{\frac{1}{4}u^2 - \frac{1}{3}u^6 + 2A}} = dt$$

Integrate both sides.

$$\int^u \frac{ds}{\sqrt{\frac{1}{4}s^2 - \frac{1}{3}s^6 + 2A}} = t + B$$

Now change back to the original variables, x and y.

$$\int^{\sqrt{x}y} \frac{ds}{\sqrt{\frac{1}{4}s^2 - \frac{1}{3}s^6 + 2A}} = \ln x + B$$